Detection model based on representation of quantum particles by classical random fields: Born's rule and beyond

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Abstract

Recently a new attempt to go beyond quantum mechanics (QM) was presented in the form of so called prequantum classical statistical field theory (PCSFT). Its main experimental prediction is violation of Born's rule which provides only an approximative description of real probabilities. We expect that it will be possible to design numerous experiments demonstrating violation of Born's rule. Moreover, recently the first experimental evidence of violation was found in the triple slits interference experiment, see [1]. Although this experimental test was motivated by another prequantum model, it can be definitely considered as at least preliminary confirmation of the main prediction of PCSFT. In our approach quantum particles are just symbolic representations of "prequantum random fields," e.g., "electron-field" or "neutron-field"; photon is associated with classical random electromagnetic field. Such prequantum fields fluctuate on time and space scales which are essentially finer than scales of QM, cf. 't Hooft's attempt to go beyond QM [2]–[4]. In this paper we elaborate a detection model in the PCSFT-framework. In this model classical random fields (corresponding to "quantum particles") interact with detectors inducing probabilities which match with Born's rule only approximately. Thus QM arises from PCSFT as an approximative theory. New tests of violation of Born's rule are proposed.

1   Introduction

Since the first days of creation of QM, its formalism and especially its interpretation have been inducing permanent debates. Bohr's postulate on completeness of QM, i.e., impossibility to go beyond it and to develop a finer description of micro-world than given by QM-formalism, was especially stimulating for these debates – starting with work of Einstein-Podolsky-Rosen [5]. In spite of so called no-go theorems – von Neumann, Bell, Kochen-Specker,... (see [7], [6] or [8] for discussions) various attempts to go beyond QM have been done during last 80 years. We can mention De Broglies' theory of double solution which was later elaborated in Bohmian mechanics, stochastic electrodynamics (SED), semiclassical model for quantum optics, Nelson's stochastic QM and its generalization by Davidson and, recently, 't Hooft's model, see, e.g., [11]–[18], [2], [3], also cf. V. I. Manko and O. V. Manko [19], [20] as well as Bracken and Wood [21], [22]; ; see also recent paper of T. Else on justification of 't Hooft's approach [23]. These models have their own advantages and disadvantages. In any event, their creation demonstrated that some possibilities to go beyond QM still can be found (in spite no-go theorems).

Recently [24] – [27] a new attempt to go beyond quantum mechanics (QM) was done in the form of so called prequantum classical statistical field theory (PCSFT). Hidden variables are given by classical fields. These prequantum fields fluctuate on time and space scales which are essentially finer than scales of QM, cf. 't Hooft's attempt to go beyond QM [2], [3]. However, opposite to Einstein, PCSFT does not assign results of quantum observations directly to hidden variables. Bohr's idea that the whole experimental arrangement should be taken into account is basic for PCSFT-based measurement theory, see De Muynck [28] or DAriano [29] for the modern presentation. This theory for measurements of positions of quantum particles as well as discrete observables is presented in this paper. Thus our theory *combines peacefully ideas of both Einstein and Bohr.* On the one hand, we deny Bohr's philosophic principle on completeness of QM. On the other hand, we deny naive Einsteinian realism – assigning results of all possible observations to a hidden variable.

We remind that the *semiclassical model* provides a simple formalism for calculation of probabilities of photon detection, see, e.g., [11]–[13]. The semiclassical model is a purely classical field model. In particular, such a fundamental of the quantum mechanical descrip-

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tion as noncommutativity is eliminated. Quantum observables are presented by classical random variables. However, the semiclassical model is applicable only to the electromagnetic field. Moreover, not all probability distributions of QM can be reproduced, see, e.g., Scully and Zubairy [11] for discussions.

We shall also consider a model with prequantum field variables. Comparing the semiclassical model with our model (which we call *prequantum classical statistical field theory* - PCSFT), we can say:

a) PCSFT is based on a different mathematical model of random field;

b) it is applicable not only to the electromagnetic field, but for any type of field;

c) it reproduces Born's rule in the general framework, hence, it reproduces any quantum probability distribution.

d) Born's rule appears as an approximative formula for calculation of real probabilities; hence, it can be violated in better designed experiments.

Regarding b): in PCSFT the basic objects are not particles, but classical fields, by b) any type of quantum particle is described by the corresponding prequantum random field. For example, we can consider the electron-field or the neutron-field.

The foundations of PCSFT were given in author's papers [24]–[27]. It was shown that quantum average[1] gives the main contribution to corresponding average with respect to classical prequantum random field. However, in general quantum and prequantum averages do not coincide. If one believe that nature is correctly described by PCSFT and not by QM (and the latter is just an *approximative model*), then one can expect to find experimental deviations from predictions of QM.

However, measurement theory based on PCSFT has not been developed in [24]– [27]. The absence of the corresponding measurement theory induced the impression that PCSFT is a kind of ontic theory which does not have a direct relation with quantum measurements. In the present paper measurement theory based on PCSFT will be presented. The basic idea is that a so called quantum state (e.g., a pure state) is simply a label for an ensemble of classical fields – *classical*

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[1]It is given by von Neumann's trace formula which is based on Born's rule.

random field. We describe the process of interaction of detectors with a random field.

First we consider the simplest model of measurement in which only the square-type nonlinearities of the prequantum field, $|\phi(x)|^2$, are taken into account.[2] In such a model [30] the probability of selection of a field is proportional to its "power" – its L_2-norm. We call this approach (quadratic) *power signal-field detection theory - PFSDT.* The power of signal (classical prequantum field) plays the crucial role in PFSDT. It will be shown that in such a detection model *Born's rule* is a consequence of the well known *Bayes formula* for conditional probability.[3]

By considering experiments in which the contribution of higher powers of the prequantum field can be taken into account, one can expect *deviation from Born's rule.* Our calculations, see section 7, provide an estimate for such deviation. Such a detection theory we will also call PFSDT (with in general nonquadratic power).

Recently the first experimental evidence of violation of Born's was found in the triple slits interference experiment, see [1]. It provides at least preliminary confirmation of the main prediction of PCSFT, see section 11 for discussion.

We remind that the dimension of the space of classical fields is infinite. This induces essential mathematical difficulties. To escape these difficulties and at the same time to illustrate all distinguishing features of our model of detection of classical fields-signals, we start with a toy model with finite-dimensional state space.

2   Finite-dimensional model

Let us consider systems – "fields" – with the state space $\mathcal{H}_n$, where $\mathcal{H}_n$ is the n-dimensional real space: $\mathcal{H}_n = \mathbf{R} \times \ldots \times \mathbf{R}$. In this model the state of a system is given by a vector $v = (v_1, \ldots, v_n)$. We set $||v||^2 =$

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[2]In PCSFT average of any prequantum random field $\phi(x, \omega)$, where ω is a random parameter, is the "zero field": $\phi_0(x) = 0$ for any $x \in \mathbf{R}^3$. Dispersion of $\phi(x, \omega)$ is very small. Therefore for "statistical majority" of realizations of $\phi(x, \omega)$ its amplitude is very small. Thus the term based on $|\phi(x)|^2$ gives the main contribution into detector's output. The contributions of higher order nonlinearities, $|\phi(x)|^n, n > 2$, are essentially smaller. The first approximation of detection is based on taking into account only quadratic nonlinearity $|\phi(x)|^2$.

[3]Here everything is classical – fields, detectors, nevertheless, the output probabilities are the same as in QM.

$\sum_{j=1}^{n} v_j^2$. It will be fruitful to use the analogy with classical signal theory. We shall also call states of systems *signals.* The Euclidean norm of a signal, $||v||^2$, we shall call *signal power.*

Suppose we have a measurement device ("detector") that produces one of the values $j = 1, ..., n$:

$$j = j(v) \tag{1}$$

for a system having the state v.

Suppose also that the probability to obtain the fixed value j_0 is proportional to $v_{j_0}^2$ and the coefficient of proportionality does not depend on j_0 (but it depends on the state v). Thus the probability that such a detector produces the result $j = j_0$ for a system with state v is given by

$$P(j = j_0|v) = k_v v_{j_0}^2. \tag{2}$$

The coefficient k_v can be found from the normalization of probability by one:

$$\sum_{j_0=1}^{n} P(j = j_0|v) = k_v \sum_{j_0=1}^{n} v_{j_0}^2 = 1. \tag{3}$$

Thus $k_v = \frac{1}{||v||^2}$ and

$$P(j = j_0|v) = \frac{v_{j_0}^2}{||v||^2}. \tag{4}$$

Consider now *complex systems combined of ensembles of "fields".* To keep closer to the analogy with QM, we can call such complex systems *"particles."* Mathematically a "particle" is described by a probability measure μ on $\mathcal{H}_n$, or by a random vector

$$\xi(\omega) = (\xi_1(\omega), \ldots, \xi_n(\omega)).$$

It is assumed that $m_\mu = \int_{\mathcal{H}_n} v d\mu(v) = 0$ (zero mean value). Its dispersion

$$\sigma^2(\mu) = \int_{\mathcal{H}_n} ||v||^2 d\mu(v) = \kappa,$$

will play an important role in our considerations.

We consider the following procedure of measurement of the quantity j for "particles":

a) first the detector selects a fixed "field" v from the ensemble representing a particle;

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b) then the detector performs measurement of $j(v)$ for this selected v (by the rule which has been already formulated).

Consider a *special procedure of selection* of a fixed "field" v from the ensemble (realization of the random vector). Our basic assumption is that a measurement device under consideration works in the following way. The probability to select a "field" with the state v from the ensemble is proportional to the *square of the norm of the state*:

$$P_\mu(v) = K_\mu \, ||v||^2 = K_\mu \sum_{j=1}^{n} v_j^2.$$

The coefficient of proportionality can be again found from the normalization of probability by one:

$$1 = \int_{\mathcal{H}_n} dP_\mu(v) = K_\mu \int_{\mathcal{H}_n} ||v||^2 d\mu(v).$$

Thus

$$K_\mu = \frac{1}{\kappa} \tag{5}$$

and

$$dP_\mu(v) = \frac{||v||^2}{\kappa} d\mu(v). \tag{6}$$

Let μ be a discrete probability measure. It is concentrated in a finite number of points $v^{(1)}, ..., v^{(m)}$ of the state space. It is determined by positive weights $\mu(v^k), k = 1, .., m$, where

$$\sum_{k=1}^{m} \mu(v^k) = 1, v^{(1)}, \ldots, v^{(m)} \in \mathcal{H}_n.$$

Now the formula (6) takes the form:

$$P_\mu(v^{(i)}) = \frac{||v^{(i)}||^2}{\kappa} \mu(v^{(i)}). \tag{7}$$

It gives the probability that a measurement device (of the class under consideration) would select (for the subsequent j-measurement) a "field" having the state $v^{(i)} = (v_1^{(i)}, ..., v_n^{(i)})$.

In the discrete case we now find the final probability to obtain the concrete result $j = j_0 : \mathbf{p}_\mu(j = j_0)$. Let us apply the Bayes formula. Probability $\mathbf{p}_\mu(j = j_0)$ is produced as the result of combination (with the aid of the Bayes rule) of two probabilities: probability $P_\mu(v^{(i)})$ to

select a "field" with the state $v = v^{(i)}$ from the ensemble of "fields" composing a "particle" and probability $P(j = j_0|v^{(i)})$ to obtain the result $j = j_0$ for measurement on a "field" with the state $v^{(i)}$:

$$\mathbf{p}_\mu(j = j_0) = \sum_i P_\mu(v^{(i)}) \, P(j = j_0|v^{(i)}). \qquad (8)$$

We put in this formula the probabilities given by (7) and (4) and obtain:

$$\mathbf{p}_\mu(j = j_0) = \frac{1}{\kappa} \sum_i |v^{(i)}_{j_0}|^2 \mu(v^{(i)}). \qquad (9)$$

We now generalize (9) to the case of continuous distribution μ. We have: $P(j = j_0|v) = \frac{v^2_{j_0}}{||v||^2}$ and $P(v) = \frac{||v||^2}{\kappa}\mu(v)$. Again by using the Bayes formula we find the probability to obtain $j = j_0$ for measurement on a system from the ensemble given by μ :

$$\mathbf{p}_\mu(j = j_0) = \int_{\mathcal{H}_n} P(j = j_0|v)dP_\mu(v) = \frac{1}{\kappa} \int v^2_{j_0} d\mu(v).$$

2.1   Pure states

Let take a vector $u \in \mathcal{H}_n$ such that $||u|| = \kappa$. Consider now a Gaussian measure $\mu \equiv \mu_u$ having the zero mean value and the covariance operator

$$C_u = u \otimes u,$$

i.e., $(C_u y, y) = (u, y)^2$. This Gaussian measure is concentrated on one dimensional subspace $L_u = \{z = cu : c \in \mathbf{R}\}$. Thus

$$d\mu_u(v) = e^{-p^2/2\kappa}dp/\sqrt{2\pi\kappa}, p = (v, w),$$

where $w = \frac{u}{\sqrt{\kappa}}$. Thus $||w|| = 1$. Then

$$\mathbf{p}_\mu(j = j_0) = \frac{1}{\kappa} \int_{\mathbf{R}^n} v^2_{j_0} d\mu_u(v) = w^2_{j_0}$$

This is nothing else than the *Born's rule* that is used in quantum mechanics. The normalized vector w can be interpreted as a "pure quantum state." In our approach it is nothing else than the symbolic representation of the Gaussian random vector with the probability distribution μ_u. We could do it the other way around – by starting directly with an arbitrary normalized vector $w \in \mathcal{H}_n$, a "pure state".

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We remark that Gaussian random vector was chosen only for simplicity. The same result is reproduced by random vector having the same covariance matrix. Detectors under consideration (quantum detectors) are not able to distinguish two random vectors with the same covariance matrix.

We also emphasize that in our approach there is no difference between "pure" and "mixed" quantum states. All quantum states are simply symbols for corresponding random vectors.

2.2 Born's rule

To write this rule similarly to original Born's formula, we consider the state space $\mathcal{H}_n$ as a space of functions $v : X_n \to \mathbf{R}$, where $X_n = \{x_1, \ldots, x_n\}$ is some discrete set. Thus, instead of the set of labels $\{j = 1, 2, \ldots, n\}$, we now consider an arbitrary discrete set X_n – the space of results of measurement. Any vector $v \equiv v(x)$. We can say that the measurement process under consideration is the X-measurement: $X = X(v)$. We have

$$\mathbf{p}_\mu(X = x) = \frac{1}{\kappa} \int v^2(x) d\mu(v)$$

or

$$\mathbf{p}_\mu(x \in I) = \frac{1}{\kappa} \sum_{X \in I} \int v^2(x) d\mu(v) = \frac{1}{\kappa} \int \sum_{x \in I} v^2(x) d\mu(v), x \in X_n, I \subset X_n.$$

In particular, for the Gaussian measure μ_u, we have:

$$\mathbf{p}_{\mu_u}(X = x) = w^2(x), \; w = \frac{u}{\sqrt{\kappa}}.$$

By choosing $n = 2k$ and considering the complex representation of the real state space $\mathcal{H}_{2k}$, namely,

$$\mathcal{H}_{2k} = \mathbf{C}^k,$$

we obtain the theory of measurement in that (as in the previous considerations):

$$P(X = x|v) = \frac{|v(x)|^2}{||v||^2}, P(v)) = \frac{||v||^2}{\kappa} \mu(v),$$

and, finally,

$$\mathbf{p}_\mu(X = x) = \frac{1}{\kappa} \int_{\mathbf{C}^n} |v(x)|^2 d\mu(v),$$

in particular,
$$\mathbf{p}_{\mu_u}(X = x) = |w(x)|^2.$$

Remark. We can consider as the state space any space $\mathcal{H}_n = T \times \ldots \times T$, where T is a number field with the valuation $| \cdot |_T$. For example, T can be chosen as the field of p-adic numbers. In this case we consider a probability μ on Q_p^n and
$$P(X = x|v) = \frac{|v(x)|_p^2}{||v||^2}, \; ||v||^2 = \sum_{k=1}^{n} |v(x_k)|_p^2.$$

We emphasize that the theory of measurement under consideration is purely classical. No noncommutative mathematics was involved in the description. The crucial point was the use of Bayes' formula (and nothing else!). From the physical viewpoint, all "nonclassical features" are induced by special functioning of measurement devices ("detectors").

3 The position measurement for the prequantum field

3.1 Classical random fields

We consider the configuration space of complex random fields: $Z = L_2(\mathbf{R}^3)$, the space of square integrable complex fields, $\phi : \mathbf{R}^3 \to \mathbf{C}$. It is endowed with the norm $\|\phi\|^2 = \int_{\mathbf{R}^3} |\phi(x)|^2 dx$. A random field is a Z-valued random variable $\phi(\omega) \in Z$. Here ω is a chance parameter. We denote by μ the probability distribution of the random variable $\phi(\omega)$. It is a probability measure on Z.[4] Of course, Z has infinite dimension, but mathematical theory of such measures is well developed. We shall proceed on the physical level of rigorousness. We shall consider only random fields with zero mean value: $E\phi(\omega) = 0$. This equality simply means that for any function $f \in Z : E \int_{\mathbf{R}^3} f(x)\overline{\phi(x,\omega)}dx = 0$. Covariance of a random field is defined as
$$C_\mu(f,g) = E\Big(\int_{\mathbf{R}^3} f(x)\overline{\phi(x,\omega)}dx \Big) \Big(\int_{\mathbf{R}^3} \overline{g(x)}\phi(x,\omega)dx \Big)$$

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[4]To be completely rigorous mathematically, we should consider a Kolmogorov probability space $(\Omega, F, \mathbf{P})$, where Ω is the space of chance parameters, F is a σ-algebra of its subsets and $\mathbf{P}$ is a probability. Then $\phi : \Omega \to Z$ is a measurable function. Its probability distribution is given by $\mu(U) = \mathbf{P}(\omega \in \Omega : \phi(\omega) \in U)$ for a Borel subset U of Z.

The corresponding operator is denoted by C_μ. We recall that the covariance operator has all main features of von Neumann's density operator (self-adjoint, positively defined, trace class), besides of normalization of the trace by 1. Dispersion is given by $\sigma^2(\mu) = E\|\phi(\omega)\|^2$. We shall consider dispersion as a parameter, say κ, of our model. We remind the following useful equality: $\kappa = \text{Tr } C_\mu$.

For each ω_0, the realization of a random field $\phi(\omega_0)$ is an L_2-function. Thus a random field can be written as a function of two variables $\phi(x, \omega), x \in \mathbf{R}^3$.

3.2 Random field-signals and position measurement

We denote by E_μ an ensemble of fields represented by a probability measure μ on Z. It is the probability distribution of a random field $\phi(\omega)$. The E_μ gives realizations of the corresponding random field. By our model each "quantum particle" is just a symbolic representation of "prequantum random field".

Let X be the position observable. This observable is considered as an observable on fields. Thus $X(\omega) = X(\phi(\omega))$ is a random variable. It takes its values in $\mathbf{R}$. Our aim is to find the probability distribution of this random variable from the probability distribution of the random field.

In accordance with measurement theory a source of identically prepared quantum particles is given. In our model this source produces a sequence of random fields: $\phi^{(1)}(x, \omega), ..., \phi^{(N)}(x, \omega)$. By performing X-measurements for this sequence of random fields we obtain its probability distribution.

In measurement theory for position we consider a random field as a *random signal* interacting with detectors. Suppose that the position observable X is given by a measurement device M_X. For example, M_X can be chosen as a collection of detectors located at all points $x \in \mathbf{R}^3$. For any point x_0, we can consider the observable X_{x_0} given by a detector $M_X(x_0)$ located at x_0. For any (sufficiently regular) set $I \subset \mathbf{R}^3$ we can consider the observable X_I given by a collection of detectors $M_X(I)$ located in the domain $I \subset \mathbf{R}^3$.

4 Power signal-field detection theory – PFSDT

In our model of detection the measurement process over a random field consists of two steps:

a) selection of a field $\phi \in E_\mu$;
b) measurement on this field: $X(\phi)$.

We assume that measurement devices (detectors) are sensitive to the *power* of the (classical) field-signal. As in classical signal theory, we define the power of the field-signal ϕ at the point x_0 as

$$\pi_2(x_0, \phi) = |\phi(x_0)|^2,$$

the field-signal power in the domain $I \subset \mathbf{R}^3$ is defined as $\pi_2(I, \phi) = \int_I |\phi(x)|^2 dx$, and finally, the total power of the field-signal ϕ is given by $\pi_2(\phi) = \|\phi\|^2 = \int_{\mathbf{R}^3} |\phi(x)|^2 dx$. We now formulate the fundamental feature of the class of detectors under consideration, namely, *sensitivity to the power of a field-signal* in the form of two postulates:

Postulate 1. *The probability P_μ to select a fixed field ϕ from the random field-signal $\phi(x, \omega)$ (the ensemble E_μ) is proportional to the total power of ϕ:*

$$dP_\mu(\phi) = K_\mu \pi_2(\phi) d\mu(\phi). \tag{10}$$

The coefficient of proportionality K_μ can be found from the normalization of probability by one: $K_\mu = \frac{1}{\int_Z \pi_2(\phi) d\mu(\phi)} = \frac{1}{\sigma^2(\mu)} = \frac{1}{\kappa}$. Thus, we get

$$dP_\mu(\phi) = \frac{\pi_2(\phi)}{\kappa} d\mu(\phi), \tag{11}$$

and, for any Borel subset $U \subset Z$, we have

$$P_\mu(\phi \in U) = \frac{1}{\kappa} \int_U \|\phi\|^2 d\mu(\phi), \tag{12}$$

or in the random field notations:

$$P_\mu(\phi \in U) = \frac{1}{\kappa} E\Big(\chi_U(\phi(\omega)) \|\phi(\omega)\|^2\Big), \tag{13}$$

where $\chi_U(\phi)$ is the characteristic function of the set U.

The selection procedure of a signal from a random field for the position measurement was formalized by Postulate 1. This postulate is intuitively attractive: more powerful signals are selected more often.

We now formalize the b-step of the X-measurement in the following form:

Postulate 2. *The probability $P(X = x_0|\phi)$ to get the result $X = x_0$ for the fixed field ϕ is proportional to the power $\pi_2(x_0, \phi)$ of this field at x_0. The coefficient of proportion does not depend on x_0, so $k(x_0|\phi) \equiv k_\phi$.*

The coefficient of proportion $k(x_0|\phi)$ can be obtained from the normalization of probability by one: $1 = \int_{\mathbf{R}^3} P(X = x|\phi)dx = k_\phi \int_{\mathbf{R}^3} |\phi(x)|^2 dx$. Thus

$$k_\phi = \frac{1}{||\phi||^2}. \tag{14}$$

The probability to get $X = x_0, x_0 \in \mathbf{R}^3$, for a random field with the probability distribution μ can be obtained by using the classical Bayes' formula:

$$\mathbf{p}_\mu(X = x_0) = \int_Z P(x_0|\phi)dP_\mu(\phi)$$

$$= \int_Z \frac{|\phi(x_0)|^2}{||\phi||^2}dP_\mu(\phi). \tag{15}$$

Thus, finally, we have:

$$\mathbf{p}_\mu(X = x_0) = \frac{1}{\kappa}\int_Z |\phi(x_0)|^2 d\mu(\phi). \tag{16}$$

Of course, $\mathbf{p}_\mu(X = x)$ should be considered as the density of probability:

$$\mathbf{p}_\mu(X \in I) = \int_I \mathbf{p}_\mu(X = x)dx$$

$$= \frac{1}{\kappa}\int_Z \left(\int_I |\phi(x)|^2 dx\right)d\mu(\phi), \tag{17}$$

where I is a Borel subset of $\mathbf{R}^3$, e.g. a cube.

5   Coupling between PFSDT and QM

To find coupling between PFSDT and the quantum formalism, we introduce projectors $\hat{I}\phi(x) = \chi_I(x)\phi(x)$, where χ_I is the characteristic function of the Borel set I.

Theorem 1. *The probability measure* $\mathbf{p}_\mu$ *can be represented in the following operator form:*

$$\mathbf{p}_\mu(I) = \mathrm{Tr}\,\hat{\rho}\,\hat{\mathrm{I}}, \tag{18}$$

where

$$\hat{\rho} = C_\mu/\kappa. \tag{19}$$

We remark that the operator $\hat{\rho}$ has all properties of the von Neumann density operator. This theorem motivates the following correspondence between classical random fields and von Neumann's density operators (quantum states):

Any classical random field induces a quantum state by mapping the field in the density operator given by (19) and vice versa.

The probability distribution $\mathbf{p}_\mu$ on $\mathbf{R}^3$ given by PFSDT coincides with the probability distribution given by QM.

We remark that the correspondence between random fields and quantum states is not one-to-one: a random field is not determined uniquely by its covariance operator. However, if one restricts considerations to only *Gaussian random fields,* then the correspondence will become one-to-one.

Let Ψ be a normalized vector – a "pure state" of QM. We consider a measure μ_Ψ on Z with zero mean value and covariance operator: $C_\Psi = \Psi \otimes \Psi$. In particular, we can choose a Gaussian measure. Then we find easily that

$$\mathbf{p}_\Psi(x) \equiv \mathbf{p}_{\mu_\Psi}(x) = |\Psi(x)|^2. \tag{20}$$

Thus

$$\mathbf{p}_\Psi(I) = \int_I |\Psi(x)|^2 dx. \tag{21}$$

This is nothing else than *Born's rule.*

We find the mean value of the position x with respect to the probability measure $\mathbf{p}_\mu$. We restrict our considerations to one dimensional case. We have:

$$< x >_{\mathbf{p}_\mu} = \int_{-\infty}^{+\infty} x d\mathbf{p}_\mu(x) = \mathrm{Tr}\,\hat{\rho}\,\hat{\mathrm{x}},$$

where $\hat{x}$ is the position operator in Scrödinger's representation of QM.

Thus PFSDT (measurement model for classical random fields) produces the same probability distributions and averages as the conventional QM model. As we will see, PFSDT provides a possibility to go beyond QM.

In appendix, we present the general scheme of representation of quantum measurement as measurements on prequantum random fields.

6 Prequantum classical statistical field theory – PCSFT

We define *"classical statistical models"* in the following way: a) physical states ϕ are represented by points of some set Y (state space); b) physical variables are represented by functions $f : Y \to \mathbf{R}$ belonging to some functional space $V(Y)$; c) statistical states are represented by probability measures on Y belonging to some class $S(Y)$; d) the average of a physical variable (which is represented by a function $f \in V(Y)$) with respect to a statistical state (which is represented by a probability measure $\mu \in S(Y)$) is given by

$$< f >_\mu \equiv \int_Y f(\phi)d\mu(\phi). \tag{22}$$

A *classical statistical model* is a pair $M = (S, V)$.

We also recall the definition of the conventional quantum statistical model with the complex Hilbert state space H_c. It is described in the following way: a) physical observables are represented by operators $\hat{A} : H_c \to H_c$ belonging to the class of continuous self-adjoint operators $L_s \equiv L_s(H_c)$; b) statistical states are represented by von Neumann density operators (the class of such operators is denoted by $D \equiv D(H_c)$); d) the average of a physical observable (which is represented by the operator $\hat{A} \in L_s(H_c)$) with respect to a statistical state (which is represented by the density operator $\hat{\rho} \in D(H_c)$) is given by von Neumann's formula:

$$< \hat{A} >_D \equiv \mathrm{Tr}\ \hat{\rho}\hat{A} \tag{23}$$

The *quantum statistical model* is the pair $N_{\mathrm{quant}} = (D, L_s)$.

We are looking for a classical statistical model $M = (S, V)$ which will provide *"dequantization"* of the quantum model $N_{\mathrm{quant}} = (D, L_s)$. By dequantization we understand constructing of a classical statistical

model such that averages given by this model can be approximated by quantum averages. Approximation is based on the asymptotic expansion of classical averages with respect to a small parameter. The main term of this expansion coincides with the corresponding quantum average. Such a classical statistical model can be called a prequantum model. It is considered as to be more fundamental than QM. The latter provides only an approximative representation of a prequantum model. Our aim is prove that a prequantum model exists.

We choose the phase space $Y = Q \times P$, where $Q = P = H$ and H is the real (separable) Hilbert space. We consider Y as the real Hilbert space with the scalar product $(\phi_1, \phi_2) = (q_1, q_2) + (p_1, p_2)$. We denote by J the symplectic operator on Y : $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let us consider the class $L_{\text{symp}}(Y)$ of bounded $\mathbf{R}$-linear operators $\hat{A} : Y \to Y$ which commute with the symplectic operator:

$$\hat{A}J = J\hat{A}. \tag{24}$$

This is a subalgebra of the algebra of bounded linear operators $L(Y)$. We also consider the space of $L_{\text{symp,s}}(Y)$ consisting of self-adjoint operators.

By using the operator J we can introduce on the phase space Y the complex structure. Here J is realized as $-i$. We denote Y endowed with this complex structure by $H_c : H_c \equiv Q \oplus iP$. We shall use it later.

Let us consider the functional space $V(Y)$ consisting of functions $f : Y \to \mathbf{R}$ such that: a) the state of vacuum is preserved[5] : $f(0) = 0$; b) f is J-invariant: $f(J\phi) = f(\phi)$; c) f can be extended to the analytic function having the exponential growth: $|f(\phi)| \le c_f e^{r_f \|\phi\|}$ for some $c_f, r_f \ge 0$. The latter condition provides the possibility to integrate such functions with respect to Gaussian measures.

The following mathematical result plays the fundamental role in establishing classical $\to$ quantum correspondence: *Let f be a smooth J-invariant function. Then $f''(0) \in L_{\text{symp,s}}(Y)$. In particular, a quadratic form is J-invariant iff it is determined by an operator belonging to $L_{\text{symp,s}}(Y)$.*

We consider the space statistical states $S^\kappa(Y)$ consisting of measures μ on Y such that: a) μ has zero mean value; b) it is a Gaussian measure; c) it is J-invariant; d) its dispersion has the magnitude κ.

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[5]The vacuum state is such a classical field which amplitude is zero at any point x.

Thus these are J-invariant Gaussian measures such that

$$\int_Y \phi d\mu(\phi) = 0 \text{ and } \sigma^2(\mu) = \int_Y \|\phi\|^2 d\mu(\phi) = \kappa, \ \kappa \to 0.$$

Such measures describe small Gaussian fluctuations.

We now consider the complex realization H_c of the phase space and the corresponding complex scalar product $< \cdot, \cdot >$. We remark that the class of operators $L_{\text{symp}}(Y)$ is mapped onto the class of $\mathbf{C}$-linear operators $L(H_c)$. We also remark that, for any $\hat{A} \in L_{\text{symp,s}}(Y)$, real and complex quadratic forms coincide: $(A\psi, \phi) =< A\psi, \phi >$. We also define for any measure its (complex)covariance operator by

$$< C_\mu y_1, y_2 >= \int < y_1, \phi >< \phi, y_2 > d\mu(\phi).$$

We consider now the one parameter family of classical statistical models:

$$M^\kappa = (S^\kappa(Y), V(Y)), \ \kappa \geq 0, \tag{25}$$

By making in the Gaussian infinite-dimensional integral the change of variables (field scaling):

$$\phi \to \phi/\sqrt{\kappa} \tag{26}$$

we obtain the following result [24]– [27]:

Let $f \in V(Y)$ and let $\mu \in S^\kappa(Y)$. Then the following asymptotic equality holds:

$$< f >_\mu = \frac{\kappa}{2} \operatorname{Tr} \hat{\rho} \, \mathrm{f}''(0) + \mathrm{O}(\kappa^2), \ \kappa \to 0, \tag{27}$$

where the operator $\hat{\rho} = C_\mu/\kappa$.

We see that the classical average (computed in the model $M^\kappa = (S^\kappa(Y), V(Y))$ by using the measure-theoretic approach) is coupled through (27) to the quantum average (computed in the model $N_{\text{quant}} = (D(H_c), L_s(H_c))$ by the von Neumann trace-formula).

The equality (27) can be used as the motivation for defining the following classical $\to$ quantum map T from the classical statistical model $M^\kappa = (S^\kappa, V)$ onto the quantum statistical model $N_{\text{quant}} = (D, L_s)$:

$$T : S^\kappa(Y) \to D(H_c), \ \ \hat{\rho} = T(\mu) = \frac{C_\mu}{\kappa} \tag{28}$$

(the Gaussian measure μ is represented by the density matrix $\hat{\rho}$ which is equal to the covariance operator of this measure normalized by κ);

$$T : V(Y) \to L_{\mathrm{s}}(H_c), \quad \hat{A} = T(f) = \frac{1}{2}f''(0). \qquad (29)$$

Our previous considerations can be presented in the following form [24]– [27]:

Beyond QM Theorem. *The one parametric family of classical statistical models $M^\kappa = (S^\kappa(Y), V(Y))$ provides dequantization of the quantum model $N_{\mathrm{quant}} = (D(H_c),\ L_{\mathrm{s}}(H_c))$ through the pair of maps (28) and (29). The classical and quantum averages are coupled by the asymptotic equality (27).*

7 Deviation from predictions of quantum mechanics

Position measurements of higher precision arise very naturally by generalization of power field-signal detection theory – PFSDT to match the general PCSFT-framework, i.e., consideration of prequantum physical variables $f(\phi)$ which are given by nonquadratic functionals of classical field. The appearance of additional terms in (27) induces deviations from predictions of QM for averages. Now we would like to find corresponding deviations for probabilities of detection, namely, deviations from *Born's rule.*

We restrict our modelling to the case of fourth order polynomials of classical fields. The main point is that, instead of the quadratic power of a field-signal ϕ given by $\pi_2(\phi) = \|\phi\|^2$, we shall consider its perturbation by integral of the fourth power of $\phi(x)$. So, we repeat the PFSDT-scheme for the position measurement in this framework.

At the moment it is not so easy to provide an adequate physical realization of theoretical measurement scheme which will provide a possibility to check our main prediction – *violation of Born's rule.* One of possibilities is based on creation of detectors which will be much more sensible to fluctuations of the prequantum random field than the present detectors. However, as was pointed by one of referees of this paper, "since a noisy detector can be made less noisy by averaging over a larger ensemble, it would seem that the detector hardware may not need to be replaced, but just the sensitivity of the experiment in which the detector is used needs to be high." It is an extremely

important point. It seems that we need not wait for new technological jumps in detectors' development. Already now one can try to design experiments which will be more sensitive to the probabilistic structure of QM.

And really such an experiment was recently performed [1] demonstrating (at least preliminary) inconsistency of the probabilistic structure of QM with experimental data. It can be considered as the first experimental confirmation of the predictions of PCSFT, see [24]–[27], [30]; see section 11 for discussion.

The measurement process over a random field again consists of two steps: a) selection of a field $\phi \in E_\mu$ from an ensemble of fields E_μ; b) measurement on this field: $X(\phi)$.

We assume that measurement devices (detectors) are sensitive to a "2+4"-*power* of the (classical) field-signal. We define this power of the field-signal ϕ at the point x_0 as

$$\pi_{2,4}(x_0, \phi) = |\phi(x_0)|^2 + |\phi(x_0)|^4,$$

the field-signal "2+4"-power in the domain $I \subset \mathbf{R}^3$ is defined as $\pi_{2,4}(I, \phi) = \int_I (|\phi(x)|^2 + |\phi(x)|^4) dx$, and finally, the total "2+4"-power of the field-signal ϕ is given by $\pi_{2,4}(\phi) = \int_{\mathbf{R}^3} (|\phi(x)|^2 + |\phi(x)|^4) dx$. We remark that, since dispersion κ of a random field $\phi(x, \omega)$ is considered as a small parameter of the model (so statistically the field is concentrated in a neighborhood of $\phi \equiv 0$), the additional perturbation term $\int_{\mathbf{R}^3} |\phi(x, \omega)|^4 \, dx$ is small from the point of view of random fluctuations.

We now formulate the fundamental feature of the class of detectors under consideration, namely, *sensitivity to the "2+4"-power of a field-signal* in the form of two postulates:

Postulate 1: "2+4"-power. *The probability P_μ to select a fixed field ϕ from the random field-signal $\phi(x, \omega)$ (the ensemble E_μ) is proportional to the total "2+4"-power of ϕ :*

$$dP_\mu(\phi) = K_\mu \pi_{2,4}(\phi) d\mu(\phi). \tag{30}$$

The coefficient of proportionality K_μ can be found from the normalization of probability by one: $K_\mu = \frac{1}{\int_Z \pi_{2,4}(\phi) d\mu(\phi)}$. Thus, we get

$$dP_\mu(\phi) = \frac{\pi_{2,4}(\phi)}{\int_Z \pi_{2,4}(\phi) d\mu(\phi)} d\mu(\phi), \tag{31}$$

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and, for any Borel subset $U \subset Z$, we have

$$P_\mu(\phi \in U) = \frac{1}{\int_Z \pi_{2,4}(\phi)d\mu(\phi)} \int_U \pi_{2,4}(\phi)d\mu(\phi), \qquad (32)$$

or in the random field notations:

$$P_\mu(\phi \in U) = \frac{1}{E\pi_{2,4}(\phi(\omega))} E\Big(\chi_U(\phi(\omega))\pi_{2,4}(\phi(\omega))\Big), \qquad (33)$$

where $\chi_U(\phi)$ is the characteristic function of the set U.

The selection procedure of a signal from a random field for the position measurement was formalized by Postulate 1. We now formalize the b-step of the X-measurement in the following form:

Postulate 2: "2+4"-power. *The probability $P(X = x_0|\phi)$ to get the result $X = x_0$ for the fixed field ϕ is proportional to the "2+4"power $\pi_{2,4}(x_0, \phi)$ of this field at the point x_0. The coefficient of proportion does not depend on x_0, so $k(x_0|\phi) \equiv k_\phi$.*

The coefficient of proportion $k(x_0|\phi)$ can be obtained from the normalization of probability by one: $1 = \int_{\mathbf{R}^3} P(X = x|\phi)dx = k_\phi \int_{\mathbf{R}^3} \pi_{2,4}(x, \phi)dx$. Thus

$$k_\phi = \frac{1}{\pi_{2,4}(\phi)}. \qquad (34)$$

The probability to get (for the position observation) the result $X = x_0, x_0 \in \mathbf{R}^3$, for a random field with the probability distribution μ can be obtained by using the classical Bayes' formula:

$$\mathbf{p}_\mu(X = x_0) = \int_Z P(x_0|\phi)dP_\mu(\phi)$$

$$= \int_Z \frac{|\phi(x_0)|^2 + |\phi(x_0)|^4}{\pi_{2,4}(\phi)}dP_\mu(\phi). \qquad (35)$$

Thus, finally, we have: $\mathbf{p}_\mu(X = x_0)$

$$= \frac{1}{\int_Z \pi_{2,4}(\phi)d\mu(\phi)} \int_Z (|\phi(x_0)|^2 + |\phi(x_0)|^4)d\mu(\phi). \qquad (36)$$

Of course, $\mathbf{p}_\mu(X = x)$ should be considered as the density of probability:

$$\mathbf{p}_\mu(X \in I) = \int_I \mathbf{p}_\mu(X = x)dx$$

$$= \frac{1}{\int_Z \pi_{2,4}(\phi)d\mu(\phi)} \int_Z \Big(\int_I (|\phi(x)|^2 + |\phi(x)|^4)dx \Big) d\mu(\phi), \qquad (37)$$

where I is a Borel subset of $\mathbf{R}^3$, e.g. a cube. It is convenient to make the field scaling (26) to move from the probability μ having dispersion κ to the corresponding normalized probability ν. By this scaling we find direct dependence of probabilities of detection on the small parameter κ. Then we can represent the coefficient of proportion as $K_\mu = \frac{1}{\kappa + \kappa^2 c_4}$, where $c_4 = \int_Z \int_{-\infty}^\infty |\phi(x)|^4 dx d\nu(\phi)$. We find the following dependence on the small parameter κ (the dispersion of random fluctuations) of the probability of the position detection:

$$\mathbf{p}_\mu(X = x_0) = \frac{1}{\kappa + \kappa^2 c_4} \int_Z (\kappa|\phi(x_0)|^2 + \kappa^2|\phi(x_0)|^4)d\nu(\phi)$$

$$= \frac{1}{1 + \kappa c_4} \int_Z (|\phi(x_0)|^2 + \kappa|\phi(x_0)|^4)d\nu(\phi). \qquad (38)$$

If in the formula (16) (for PFSDT with quadratic power) we make scaling (26), we obtain:

$$\mathbf{p}_\mu(X = x_0) = \int_Z |\phi(x_0)|^2 d\nu(\phi). \qquad (39)$$

The same result we obtain by the considering the limit $\kappa \to 0$ in (38). Thus the model presented in this section is really the $O(\kappa)$ perturbation of the model considered in section 3 (and hence of QM).

We come back to our model of detection taking into account the fourth power of the signal-field: $\mathbf{p}_\mu(X \in I)$

$$= \frac{\int_Z \int_I |\phi(x)|^2 dx d\nu(\phi) + \kappa \int_Z \int_I |\phi(x)|^4 dx d\nu(\phi)}{1 + \kappa c_4}.$$

Hence $\mathbf{p}_\mu(X \in I)$

$$\approx (1 - \kappa c_4)\Big(\int_Z \int_I |\phi(x))|^2 dx d\nu(\phi) + \kappa \int_Z \int_I |\phi(x)|^4 dx)d\nu(\phi) \Big)$$

$$= \int_Z \int_I |\phi(x)|^2 dx d\nu(\phi)) + \kappa \Big[\int_Z \int_I |\phi(x)|^4 dx d\nu(\phi)$$

$$- \int_Z \int_I |\phi(x)|^2 dx d\nu(\phi) \times \int_Z \int_{\mathbf{R}^3} |\phi(x)|^4 dx d\nu(\phi) \Big].$$

The first summand gives the well known Born's rule (the conventional QM-prediction). The second summand (we denote it by $\Delta(I, \mu, \kappa)$) gives the deviation from the Born's rule.

We consider now this deviation in the case of so called pure state $\Psi, ||\Psi|| = 1$. In our approach this corresponds to the case when the normalized measure ν is the Gaussian measure $\nu \equiv \nu_\Psi$ with the covariance operator: $C = \Psi \otimes \Psi$. We have $\Delta(I, \Psi, \kappa)$

$$= \kappa \Big[\int_I |\Psi(x)|^4 dx - \int_I |\Psi(x)|^2 dx \int_{\mathbf{R}^3} |\Psi(x)|^4 dx \Big] \qquad (40)$$

Suppose[6] that supp $\Psi \subset I$, so the wave function is zero outside the set I. Then $\Delta \equiv 0$.

Further we consider one dimensional case. Let now $\Psi(x) = H, L/2 \le x \le L/2$. Thus $H^2 L = 1$, so $L = 1/H^2$. We choose $I = [0, L/2]$:

$$\int_I |\Psi(x)|^2 dx = 1/2, \int_{\mathbf{R}^3} |\Psi(x)|^4 dx = H^4 L = H^2,$$

and

$$\int_I |\Psi(x)|^4 dx = \frac{H^4 L}{2} = \frac{H^2}{2}, \ \Delta = \kappa(\frac{H^2}{2} - \frac{H^2}{2}) = 0.$$

This calculation gave a hint that an asymmetric probability distribution may induce nontrivial Δ. We choose:

$$\Psi(x) = \left\{ \begin{array}{l} H, \ -L/2 \le x \le 0 \\ kH, \ 0 < x \le L/2 \end{array} \right.$$

Hence, $1 = ||\Psi||^2 = LH^2(k^2 + 1)/2$, so $L = 2/(H^2(k^2 + 1))$. Here $I = [L/2, 0], \int_I |\Psi(x)|^2 dx = H^2 L/2 = 1/(k^2 + 1)$;

$$\int_{\mathbf{R}^3} |\Psi(x)|^4 dx = \Big(\frac{1 + k^4}{1 + k^2}\Big) H^2, \ \int_I |\Psi(x)|^4 dx = \frac{H^2}{k^2 + 1}.$$

$$\Delta = \frac{\kappa H^2 k^2 (1 - k^2)}{(1 + k^2)^2}.$$

If $k > 1$, then $\Delta(I, \Psi, \kappa) < 0$. Suppose that H increases (and k is fixed) then the deviation from Born's rule will be always negative and this deviation will be increasing. So, for large H, the probability to find a system in I will be essentially less than predicted by QM. For example, choose $k = 2$, then

$$\Delta = 0,48\kappa H^2.$$

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[6]To be mathematically rigorous, we consider $\Psi \in L_{2,4}(\mathbf{R}^3)$: both integrals $\int |\phi(x)|^2 dx$ and $\int |\phi(x)|^4 dx$ are finite.

On the other hand, by choosing $k < 1$, we shall get the positive deviation. For $k = 1$, we have $\Delta = 0$ and there will be no deviation from Born's rule.

All our considerations are purely qualitative, since we do not know the magnitude of κ. But one may expect that such a qualitative effect as decreasing and increasing the probability (comparing with the Born's rule) can be observed in experiments.

8   Averages

We take a nonquadratic functional of classical fields:

$$f_x(\phi) = \int_{-\infty}^{+\infty} x(|\phi(x)|^2 + |\phi(x)|^4)dx.$$

It is mapped onto the position operator by the map T : PCSFT$\rightarrow$ QM. PCSFT gives the following average:

$$< f_x >_\mu = \int_{-\infty}^{+\infty} x \int_Z (\phi(x)|^2 + |\phi(x)|^4)d\mu(\phi)dx$$

$$= \kappa \int_{-\infty}^{+\infty} x \int_Z |\phi(x)|^2 d\nu(\phi)dx$$

$$+\kappa^2 \int_{-\infty}^{+\infty} x \int_Z |\phi(x)|^4 d\nu(\phi)dx.$$

On the other hand, by using the distribution provided by PFSDT (field power detection model) we get:

$$< x >_{\mathbf{p}_\mu} = \int_{-\infty}^{+\infty} x d\mathbf{p}_\mu(x)$$

$$= \left[\kappa \int_{-\infty}^{+\infty} x \int_Z |\phi(x)|^2 d\nu(\phi)dx\right.$$

$$+\kappa^2 \int_{-\infty}^{+\infty} x \int_Z |\phi(x)|^4 d\nu(\phi)dx\right]$$

$$/\left[\kappa + \kappa^2 \int_{-\infty}^{+\infty} \int_Z |\phi(x)|^4 d\nu(\phi)dx\right].$$

We consider the normalization based on the field-power functional $\pi_{2,4}(\phi)$. Then

$$\frac{< f >_\mu}{< \pi_{2,4} >_\mu} = \int_{-\infty}^{+\infty} x d\mathbf{p}_\mu(x). \qquad (41)$$

Thus by normalizing the PCSFT-average by such a field-power functional we obtain the quantity which coincides with PFSDT-average. The PCSFT-averages should be normalized in the corresponding way to obtain PFSDT-averages. Of course, this is valid only for polynomial functionals of the same field-power as detectors considered by PFSDT.

We now consider the field-power functional $\pi_2(\phi) = ||\phi||^2$. The basic asymptotic equality of PCSFT can be written in the form:

$$\frac{< f >_\mu}{< \pi_2 >_\mu} = < T(f) >_{T(\mu)} + O(\kappa), \kappa \to 0.$$

9   PFSDT: scheme for local measurements

To obtain a local measurement scheme, we should take into account that any detector is located in a special domain, say O, of space. More generally, if a detector measures some variable, e.g., energy, it operates only in a special range of variation of this variable. This fundamental fact of measurement theory should be taken into account, cf. Haag [31] and especially [32], and our PFSDT should be modified. We consider again position measurement. As before, measurement process over a random field consists of two steps: a) selection of a fixed field ϕ from the random prequantum field; b) measurement on this field.

Thus we proceed under the assumption that the detector operates in the domain O.

Postulate 1. (Local) *Probability to select a fixed field ϕ from the random field-signal $\phi(x, \omega)$ is proportional to the power of the field-signal ϕ in the domain O : $dP_\mu(\phi|O) = K_{\mu,O} \pi_2(O; \phi) d\mu(\phi)$.*

The coefficient of proportionality K_μ can be found from the normalization of probability by one: $K_{\mu,O} = \frac{1}{\kappa_O}$, where $\kappa_O = \int_Z \pi_2(O; \phi) d\mu(\phi)$. Thus, we get $dP_\mu(\phi|O) = \frac{\pi_2(O;\phi)}{\kappa_O} d\mu(\phi)$, Postulate 1 is intuitively attractive: signals which are more powerful in the domain of detection O are selected more often. We now again formalize the b-step:

Postulate 2. (Local) *Probability $P(X = x_0|\phi, O)$ to get the result $X = x_0$, where $x_0 \in O$, for the fixed field ϕ is proportional to the power $\pi_2(x_0, \phi)$ of this field at x_0. The coefficient of proportion does not depend on x_0, so $k(x_0|\phi) \equiv k_{\phi,O}$.*

This coefficient can be obtained from the normalization of probability by one: $1 = \int_O P(X = x|\phi, O)dx = k_{\phi,O} \int_O |\phi(x)|^2 dx$. Thus $k_{\phi,O} = \frac{1}{\pi_2(O;\phi)}$.

The probability to get $X = x_0, x_0 \in O$, for a random field with the probability distribution μ can be again obtained by using the classical Bayes' formula: $\mathbf{p}_\mu(X = x_0|O) = \int_Z P(x_0|\phi, O)dP_\mu(\phi|O)$ $\int_Z \frac{|\phi(x_0)|^2}{\pi_2(O;\phi)}dP_\mu(\phi|O)$. Thus, finally, we have:

$$\mathbf{p}_\mu(X = x_0|O) = \frac{1}{\kappa_O} \int_Z |\phi(x_0)|^2 d\mu(\phi). \tag{42}$$

Of course, $\mathbf{p}_\mu(X = x|O)$ should be considered as the density of probability. For a domain $I \subset O$,

$$\mathbf{p}_\mu(X \in I|O) = \int_I \mathbf{p}_\mu(X = x|O)dx$$

$$= \frac{1}{\kappa_O} \int_Z \Big(\int_I |\phi(x)|^2 dx \Big) d\mu(\phi). \tag{43}$$

10   Double clicks problem

One of the referees presented the following important objection to the presented model, PFSDT, of detection for PCSFT:

"As there are no point particles explicitly built into this model, the issue of locality is potentially a concern. For example, consider two detectors A and B located in volumes O_A and O_B which are sufficiently far from each other and consider a single particle wave function whose support simultaneously includes the volumes of both detectors at the time of measurement as the wave function moves through both detectors. Assume ideal detectors so that their quantum efficiency is 100% and there are no false counts. According to quantum mechanics, if one detector finds a particle then the other detector can not find one. The conditional probability that a particle is found at B given that one was found at A must vanish. This is a statement of locality together with the property of a particle. A single particle

can not be detected in two two spacelike separated 4-volumes. This gedanken experiment would seem to cause a problem for a classical field model. I would expect the classical field for a pure state also to be non-zero in the volumes of both detectors. So the probability of detecting simultaneously a particle at detector A and at detector B as predicted by the classical field model would be non-zero in contradiction to quantum mechanics. I think the author should address this issue. Perhaps the current model is limited to the behavior of a single detector or to mutiple detectors in high flux beams, but is not applicable to multiple detectors with single particle quantum states (or very low flux beams). This would mean that it is a semiclassical model with a certain domain of applicability, but it is not completely equivalent to quantum mechanics."

Of course, the model presented in this paper was elaborated for only one detector measurement. The problem of its extension to measurements performed simultaneously by a few detectors has not been yet solved, including, generalization to multi-particle systems – prequantum random fields representing entangled quantum particles. However, even in the presented framework one can see that the problem mentioned by the referee could not be ignored. The simplest solution is to agree (at leat5 at the moment) with the referee and consider PFSDT as a semiclassical model, i.e., without to pretend to cover QM completely. It might be a right decision, especially since, as it was remarked, "multi-particle systems" (e.g., random fields corresponding to entangled systems) have not been handled. Nevertheless, I would like to point to some similarity of this problem with the problem of experimental testing of violation of Bell's inequality. In the latter case any local model with hidden variables evidently confronts with the *theoretical formalism of QM*. However, the experimental situation is extremely complicated. Various loopholes do not give a possibility to be completely sure that, e.g., Aspect's experiment [35] or Weihs-Zeilinger's experiment [33], [34] provided the final confirmation of Bell's argument, see [36]–[39] for debates.

I suspect that similar loopholes will appear in the experimental test considered by the referee. Unfortunately, I do not know so well the experimental situation in this domain (at least comparing with Bell's test). My first reflection is that it is not easy to prepare a single system (which is sufficiently massive) in e.g. *camel type state* – with two domains of essential concentration of the wave function which are sufficiently far from each other (comparing with the velocity of light).

[7] If it is really the case then one should work with the camel-like states of photons. However, problems of Bell's test will also appear in this situation, e.g., the problem of efficiency of detectors or the time window problem, cf. [34].

By using loopholes one may speculate that the "single detector click" prediction of QM is also an idealization which might be violated in better experiments. However, at the moment it is too early to present such a conjecture in the PFSDT/PCSFT-framework, especially in the absence of a few detectors generalization of the presented measurement theory. I just remark that the famous experiment of Grangier [41], [40] testing the semiclassical model can be easily objected on the basis of the detection loophole (in the same way as Aspect's type experiments). Moreover, one can easily present a model with hidden variables which reproduce Grangier's experimental data [42], [43] (e.g., by playing with discrimination thresholds).

11 Discussion on the triple slit experiment

We remark that the original motivation for experiment done in [1] was based on Sorkin's works on the *"sum over histories"* approach to QM, [44]. The latter is also a kind of prequantum model. In Sorkin's approach quantum probabilities are embedded in theory of generalized "probabilities" (the latter have unusual properties, it seems that the standard frequency interpretation is impossible). Sorkin's model is essentially more general than QM. In principle, a model which matches with QM for the two slit interference, but not for the triple slit can be considered. In such a model Born's rule should be violated, otherwise there would be no difference even in the case of the triple slit experiment. This motivated Sorkin to propose a test for violation of Born's rule based on the triple slit experiment, see [1].

PSCFT differs crucially from Sorkin's model [44]. It is based on ordinary (Kolmogorov) ensemble probability. Thus we need not appeal to generalized probabilistic models to get quantum probabilities, cf. Sorkin [44] and also Khrennikov [45]. Moreover, by PCSFT there

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[7]We remind that in PCSFT pure quantum states are just labels for classical prequantum random fields. However, the support of a "quantum wave function" coincides with the support of realizations of the corresponding prequantum random field.

are no reasons to expect violation of Born's rule only in the triple (or more) slit experiment. In principle, violation is expected already in the two slit experiment. In one of future papers a comparative analysis of two and triple slit experiments from the point of view of PCSFT will be performed. Such an analysis may be useful to find experimental designs which are most sensitive to perturbation of Born's rule (from the viewpoint of PCSFT).

However, even without such an analysis it is clear that the triple slit experiment can be considered as increasing of precision of the position measurement comparing with the two slit experiment. In accordance with our model higher precision measurement of position may produce deviation from Born's rule. From this viewpoint the n-slit experiment may produce more visible deviation for large n.

12 Other tests of Born's rule

Let $a = \pm 1$ and $b = \pm 1$ be two observables which are presented in QM by two self-adjoint operators, $\widehat{a}, \widehat{b}$. In the case of nondegenerate spectra, they are realized in the two dimensional complex space. Consider their eigenvectors $\widehat{a}e_{\pm}^a = \pm e_{\pm}^a, \widehat{b}e_{\pm}^b = \pm e_{\pm}^b$.

12.1 Double stochasticity test

Consider the matrix of transition probabilities $P^{b|a}$:

$$p_{\beta|\alpha} \equiv \mathbf{P}(b = \beta|a = \alpha),$$

where $\alpha, \beta = \pm 1$.

Both in classical and quantum probabilistic models this matrix is always *left stochastic.* A left stochastic matrix is a square matrix whose columns consist of nonnegative real numbers whose sum is 1. Really, for each value $a = \alpha$,

$$\sum_{\beta} \mathbf{P}(b = \beta|a = \alpha) = 1.$$

However, in QM, unlike the classical model, this matrix is *always* doubly stochastic.

We remind that in a *doubly stochastic matrix* all entries are nonnegative and all rows and all columns sum to 1: for each value $b = \beta$,

$$\sum_{\alpha} \mathbf{P}(b = \beta|a = \alpha) = 1. \tag{44}$$

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It is a simple consequence of Born's rule, see [46] for details.

Any *experimental test of double stochasticity* can be considered as a test of validity of Born's rule. If one finds two obseravbles such that (44) is violated, then the formalism of QM, in particular, Born's rule cannot be applied.

We remark that, unlike Sorkin's test, the aim of the double stochasticity test is not to distinguish QM from a more general nonclassical model. In the classical model the matrix of transition probabilities can be doubly as well non-doubly stochastic. One of consequences of this remark is that the double stochasticity test is most useful for massive particles (since double stochasticity takes place for classical electromagnetic field, e.g., in classical optics).

I propose to check carefully double stochasticity for spin projections on two axes a and b, e.g., for electrons. First the spin projection on the axis a is measured in the form of nondestructive filtration. After the Stern-Gerlach magnet with the a-orientation, electrons with spins up ($a = +1$) and down ($a = -1$) go to different directions. The second Stern-Gerlach magnet (with the b-orientation) is used for the final measurement. Transition probabilities are calculated, They are summed up with respect to $\alpha = \pm$ for each $b = \beta$.

12.2   Interference magnitude test

This test is more complicated, since it involves preparation of a variety of states, i.e., it should be repeated for ensembles of "particles" in different states. Consider again two observables $a = \pm 1$ and $b = \pm 1$. Take any pure state ψ. Then it is easy to derive the following *formula of interference of probabilities*, see [46]:

$$\mathbf{P}_\psi(b = \beta) = \sum_\alpha \mathbf{P}_\psi(a = \alpha)\mathbf{P}(b = \beta|a = \alpha) \tag{45}$$

$$+2\cos\theta\sqrt{\mathbf{P}_\psi(a = \alpha_1)\mathbf{P}(b = \beta|a = \alpha_1)\mathbf{P}_\psi(a = \alpha_2)\mathbf{P}(b = \beta|a = \alpha_2)},$$

where $\alpha_1 = +1, \alpha_2 = -1$.

Here $\mathbf{P}_\psi(b = \beta) = |\langle\psi, e_\beta^b\rangle|^2, \mathbf{P}_\psi(a = \alpha) = |\langle\psi, e_\alpha^a\rangle|^2, \mathbf{P}(b = \beta|a = \alpha_1) = |\langle e_\beta^b, e_\alpha^a\rangle|^2$. The phase θ can be found from phases of considered scalar products. We remark that (45) implies the following inequality

$$\left|\frac{\mathbf{P}_\psi(b = \beta) - \sum_\alpha \mathbf{P}_\psi(a = \alpha)\mathbf{P}(b = \beta|a = \alpha)}{2\sqrt{\mathbf{P}_\psi(a = \alpha_1)\mathbf{P}(b = \beta|a = \alpha_1)\mathbf{P}_\psi(a = \alpha_2)\mathbf{P}(b = \beta|a = \alpha_2)}}\right| \leq 1. \tag{46}$$

Since all probabilities involved in this inequality can be estimated by experimental frequencies, this condition can be checked experimentally. Any statistical essential deviation from this inequality can be considered as a sign of violation of Born's rule (which was basic in the derivation of the formula of interference of probabilities). Of course, the main problem is to find an appropriative pair of observables and a preparation procedure for systems ("pure state" ψ.)

Consider e.g. the case such that all transition probabilities are equal to 1/2. Set $p_+ = \mathbf{P}_\psi(b = +)$ and $q_+ = \mathbf{P}_\psi(a = +)$. Then (46) is simplified

$$|p_+ - 0.5| > \sqrt{q_+(1 - q_+)}. \tag{47}$$

Thus both variables should be essentially nonsymmetrically distributed.

13   Appendix: Measurement of observables with discrete spectra

Let $f(\phi)$ be a classical physical variable (functional of classical fields). Consider corresponding operator A, see (29). By our model quantum observable represented by the operator A performs an approximative measurement of the classical variable f. For simplicity, suppose that A has purely discrete spectrum. By our prequantum model detectors described by the formalism of QM work in the following way:

We define the A-power of the field-signal $\phi(x)$ at the point $a \in \mathbf{R}$ as $\pi_2^{(A)}(a, \phi) = |<\phi, e_a>|^2$, where $Ae_a = ae_a$; on the interval $I \subset \mathbf{R}$ as $\pi_2^{(A)}(I, \phi) = \sum_{a \in I} \pi_2^A(a, \phi)$ and the total power as $\pi_2^{(A)}(\phi) = \sum_{a \in \mathbf{R}} \pi_2^A(a, \phi) \equiv ||\phi||^2$. We repeat shortly our scheme of measurement based on interaction of an A-detector with a prequantum classical field (paying the role of signal):

a) selection of the concrete field signal ϕ with probability proportional to the square of norm;

b) production of some value $A = a$ with probability proportional to the A-power of the prequantum field-signal ϕ at the point a.

As in previous considerations, these postulates and the conventional Bayes' formula imply:

$\mathbf{p}_\mu^{(A)}(a) = \frac{1}{\kappa} \int_Z \pi_2^{(A)}(a, \phi) d\mu(\phi) = \frac{1}{\kappa} \int_Z |<\phi, e_j>|^2 d\mu(\phi)$.

Let now $\mu = \mu_\Psi, ||\Psi|| = 1$. It is Gaussian measure with zero mean value and covariance $C = \Psi \otimes \Psi$. Then $\int_Z |<\phi, e_a>|^2 d\mu_\Psi(\phi) = |<\Psi, e_a>|^2$. Hence $\mathbf{p}_{\mu_\Psi}^{(A)}(a) = |<\Psi, e_a>|^2$. This is Born's rule.

A local modification of this model – as in section 9 – is evident.

14 Appendix: Wiener-Siegel differential space theory

There is some similarity between PFSDT/PCSFT theory and the differential space theory of Wiener and Siegel [47], [48] which was made more palatable for physicists by Bohm and Bub (WSBB, [49]– [51]). Both theories have a random classical field. In WSBB, there is actually a random Hilbert space vector, but in the position basis this becomes a random field. WSBB uses a universal algorithm for finding the value of a quantum observable given the quantum state vector and the particular instance of the classical field, called the polychotomic algorithm. This avoids the locality problem mentioned in section 10, because it ensures that the particle (for a single particle wave function) can only be found at one point at one time. But the polychotomic algorithm is not local to one detector, but rather selects a position coordinate for the particle using the whole wave function and the whole classical random field.

We remark that the PFSDT algorithm presented in section 3 is neither local, since at the first step of its realization a detector takes into account the total power of the classical field – its L_2-norm (or integrals with higher nonlinearities in more general framework). However, the algorithm is easily transformed into a local detection algorithm, see section 9 by using Haag's approach. However, the problem of double clicks arises.

More detailed comparing of PFSDT/PCSFT with WSBB will be presented in one of future publications.

Conclusion. *A model of measurement in which "quantum particles" are represented by classical random fields interacting with detectors is presented. The main prediction of this model is deviation from the basic probabilistic law of QM, Born's rule. The latter is only an approximative formula. We expect that numerous experiments demonstrating violation of Born's rule will be designed. Our prediction has been already confirmed (at least preliminary) in the experiment [1].*

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